Section 1. Definitions and Per Unit System

Definitions related to power. RMS values, active and reactive power, power factor, power flow in transmission lines. Review of per unit system and the advantages that it offers.

1.1 Single-Phase Definitions

The root-mean-squared (RMS) value of a periodic voltage (or current) waveform is

$$V_{RMS} = \sqrt{\frac{1}{T}\int_{t}^{t+T} v^2(t)dt}$$, where T is the period of $v(t)$.

If $v(t) = V\sin(\omega t + \delta)$, where V is the peak value, then using $\sin^2(A) = \frac{1-\cos(2A)}{2}$, V_{RMS} becomes

$$V_{RMS} = \frac{V}{\sqrt{2}} .$$

Instantaneous power flowing to a load, using the sign convention shown in Figure 1.1, is defined as

$$p(t) = v(t) \bullet i(t) .$$

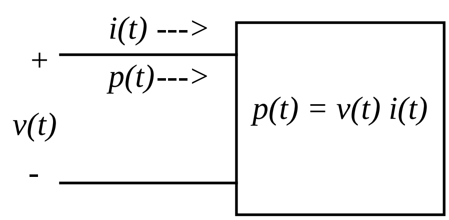

Figure 1.1: Instantaneous Power Flowing Into a Load

Average power flowing to a load is defined as

$$P = \frac{1}{T}\int_{t}^{t+T} p(t)dt$$, where T is the period of $p(t)$.

If $v(t) = V\sin(\omega t + \delta)$, $i(t) = I\sin(\omega t + \theta)$, then the instantaneous power becomes

$$p(t) = v(t) \bullet i(t) = \frac{VI}{2}\left[\cos(\delta - \theta) - \cos(2\omega t + \delta + \theta)\right].$$

Note that $p(t)$ has double-frequency and time-invariant components. The average value of $p(t)$ is the time-invariant component, or

$$P = \frac{VI}{2}\cos(\delta - \theta) \ .$$

Since $V_{RMS} = \frac{V}{\sqrt{2}}$ and $I_{RMS} = \frac{I}{\sqrt{2}}$, then

$$P = V_{RMS} I_{RMS} \cos(\delta - \theta) \ ,$$

where $\cos(\delta - \theta)$ is known as the displacement power factor (pf).

Reactive power Q is defined as

$$Q = V_{RMS} I_{RMS} \sin(\delta - \theta) \ .$$

Since $\sin^2(x) + \cos^2(x) = 1$, then

$$P^2 + Q^2 = V_{RMS}^2 \bullet I_{RMS}^2 \ .$$

Complex power S is defined as

$$S = P + jQ \ ,$$

so that

$$P = \mathrm{Real}\{S\},\ Q = \mathrm{Imag}\{S\} .$$

The magnitude of S is

$$|S| = \sqrt{P^2 + Q^2} \ ,$$

which is identical to

$$|S| = V_{RMS} \bullet I_{RMS} \ .$$

Using voltage and current phasors $\tilde{V} = |V_{RMS}| \angle\delta$ and $\tilde{I} = |I_{RMS}| \angle\theta$, the product $\tilde{V}\tilde{I}^*$ is

$$|V_{RMS}| \bullet |I_{RMS}| \angle(\delta - \theta) = |V_{RMS}| \bullet |I_{RMS}| \cos(\delta - \theta) + j|V_{RMS}| \bullet |I_{RMS}| \sin(\delta - \theta) \ .$$

Therefore,

$$S = \tilde{P} + jQ = \tilde{V}\tilde{I}^* \ .$$

When $\tilde{I}$ lags $\tilde{V}$, Q is positive, and the power factor is lagging. When $\tilde{I}$ leads $\tilde{V}$, Q is negative, and the power factor is leading. Thus, an inductive load has a lagging power factor and absorbs Q, while a capacitive load has a leading power factor and produces Q.

The total power factor *pf* is defined as

$$pf = \frac{P}{S} \ .$$

For sinusoidal systems, total power factor is identical to displacement power factor defined previously as the cosine of the relative phase angle between voltage and current.

If P and *pf* are given, Q can be calculated using

$$Q^2 = S^2 - P^2 = \frac{P^2}{pf^2} - P^2 = P^2\left(\frac{1}{pf^2} - 1\right), \text{ or}$$

$$Q = P\sqrt{\frac{1}{pf^2} - 1} \ .$$

The relationships among P, Q, S, and *pf* are shown in the power factor triangle given in Figure 1.2.

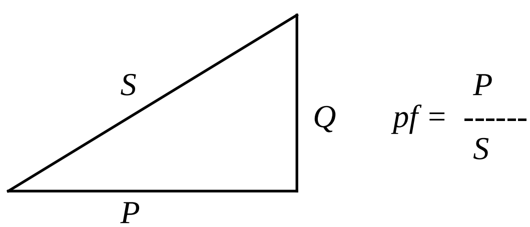

Figure 1.2: Power Factor Triangle

The impact of power factor on the Q/P ratio is given below in the Table 1.1.

pf	Q / P
1.0	0.00
0.9	0.48
0.8	0.75
0.707	1.00
0.6	1.33

Table 1.1: Impact of Power Factor on Reactive Power *Q*

Now, consider the power flow through a purely inductive circuit element, such as a lossless transmission line or transformer shown in Figure 1.3.

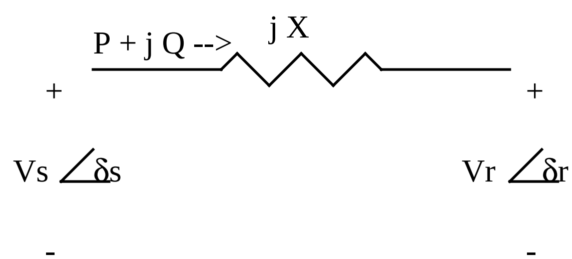

Figure 1.3: Power Flow Through a Purely Inductive Circuit Element

The active and reactive power flows, measured at the sending end S, can be shown to be

$$P = \frac{V_S V_R}{X} \sin(\theta_S - \theta_R)\,,\; Q = \frac{V_S}{X}\left[V_S - V_R \cos(\theta_S - \theta_R)\right].$$

Usually, $(\theta_S - \theta_R)$ is small, so that

$$P \,\alpha\, (\theta_S - \theta_R)\,,\; Q \,\alpha\, \frac{V_S}{X}\left[V_S - V_R\right].$$

Therefore, in inductive circuit elements, *P* tends to be proportional to voltage angle difference, and *Q* tends to be proportional to voltage magnitude difference.

1.2 Three-Phase Definitions

In a three-phase system, the total power (instantaneous, average, or complex) is the sum of the powers of the three individual phases, or

$p(t) = p_a(t) + p_b(t) + p_c(t)$, instantaneous power,

$P = P_a + P_b + P_c$, average power,

$S = (P_a + jQ_a) + (P_b + jQ_b) + (P_c + jQ_c)$, complex power.

The voltages must be measured with respect to a common reference point. In a four-wire system, the common reference should be the neutral wire (or ground). In a three-wire system, the common reference can be a derived neutral, or one of the phases (in which case the power on the reference phase is zero since the corresponding voltage is zero).

1.3 Single-Phase Per Unit System

Advantages of the per unit system:

1. Transformers can be replaced by their equivalent series impedances.
2. Equipment impedances can be easily estimated since their per unit impedances lie within a relatively narrow range.

Define four quantities in per unit of their respective base values:

$$V_{pu} = \frac{V}{V_{base}} \frac{volts}{volts} , \qquad I_{pu} = \frac{I}{I_{base}} \frac{amps}{amps} ,$$

$$S_{pu} = \frac{S}{S_{base}} \frac{voltamps}{voltamps} , \qquad Z_{pu} = \frac{Z}{Z_{base}} \frac{ohms}{ohms} .$$

The relationships among the bases are

$$S_{base} = V_{base} \bullet I_{base} \text{ , and } Z_{base} = \frac{V_{base}}{I_{base}} .$$

Once two base variables are specified, the other two base variables may be calculated.

A convenient relation, derived from the two above equations, is

$$S_{base} = V_{base} \bullet I_{base} = V_{base} \bullet \frac{V_{base}}{Z_{base}} = \frac{V_{base}^2}{Z_{base}} .$$

Consider a transformer with an $N_1 : N_2$ turns ratio and series impedance, reflected on side 1, equal Z_{L1} . Z_{L1} can be reflected to side 2 using

$$Z_{L2} = \left[\frac{N_2}{N_1}\right]^2 Z_{L1} \ .$$

Let side 1 and side 2 have base values designated by subscripts S1 and S2. Then

$$Z_{base1} = \frac{V_{base1}^2}{S_{base1}} \ , \qquad Z_{base2} = \frac{V_{base2}^2}{S_{base2}} \ ,$$

Expressing the transformer impedance on the two respective bases yields

$$Z_{L1PU} = \frac{Z_{L1}S_{base1}}{V_{base1}^2} \ , \qquad Z_{L2PU} = \frac{Z_{L2}S_{base2}}{V_{base2}^2} \ .$$

If $S_{B1} = S_{B2}$, the two above equations may be combined so that

$$Z_{L2PU} = \left[\frac{N_2}{N_1}\right]^2 \left[\frac{V_{base1}}{V_{base2}}\right]^2 Z_{L1PU} \ .$$

Substituting the relation between Z_{L1} and Z_{L2} yields

$$Z_{L2PU} = \left[\frac{N_2}{N_1}\right]^2 \left[\frac{V_{base1}}{V_{base2}}\right]^2 Z_{L1PU} \ .$$

Therefore, if $\frac{V_{base2}}{V_{base1}} = \frac{N_2}{N_1}$, then $Z_{L2PU} = Z_{L1PU}$.

Hence, if a common voltampere base is chosen on both sides of the transformer, and if the voltage bases are chosen so that they vary according to the transformer turns ratio, then the per unit series impedance of the transformer is the same value on both sides.

When analyzing a circuit with many transformers, a common voltampere base should be chosen throughout the circuit, and a voltage base should be chosen at one location. The voltage base must vary across the circuit according to the transformer turns ratios.

When analyzing a circuit in per unit, if the bases are chosen according to the above rules, transformers can be replaced by their equivalent per unit series impedances, and their turns can be ignored.

A manufacturer usually provides the impedance of a transformer on the transformer's rated voltage and power bases. However, when solving a power network circuit, the power and voltage bases must vary according to the above rules, and they may not equal the manufacturer-specified bases. Per unit impedances, specified on one base, may be converted to a new base as follows:

Given $Z_{PU,old} = \dfrac{Z}{Z_{base,old}}$, on bases $V_{base,old}$ and $S_{base,old}$, $Z_{PU,new}$ on new bases $V_{base,new}$ and $S_{base,new}$ is

$$Z_{PU,new} = \frac{Z_{ohms}}{Z_{base,new}} = \frac{Z_{PU,old} Z_{base,old}}{Z_{base,new}} = Z_{PU,old} \left[\frac{V_{base,old}^2}{S_{base,old}} \right] \left[\frac{S_{base,new}}{V_{base,new}^2} \right] .$$

1.4 Three-Phase Per Unit System

The same advantages apply to a three-phase system if the following rules are obeyed:

1. A common three-phase voltampere base is used throughout the system, where

$$S_{base,3\Phi} = 3S_{base,1\Phi} .$$

2. Once selected at a point in the network, the three-phase voltage base must vary according to the line-to-line transformer turns ratios.

Convenient formulas relating single-phase to three-phase bases are given below.

$$S_{base,1\Phi} = V_{base,line-neutral} \bullet I_{base} ,$$

$$S_{base,3\Phi} = 3S_{base,1\Phi} ,$$

$$Z_{base} = \frac{V_{base,line-neutral}^2}{S_{base,1\Phi}} = \frac{\left[V_{base,line-line} / \sqrt{3} \right]^2}{S_{base,3\Phi} / 3} = \frac{V_{base,line-line}^2}{S_{base,3\Phi}} .$$