Section 2. Symmetrical Components

Transformation matrices and the decoupling that occurs in balanced three-phase systems. Physical significance of zero sequence.

2.1 Transformation Matrix

Fortescue's Theorem: An unbalanced set of *N* related phasors can be resolved into *N* systems of phasors called the symmetrical components of the original phasors. For a three-phase system (i.e. *N* = 3), the three sets are:

1. Positive Sequence - three phasors, equal in magnitude, 120^o apart, with the same sequence (a-b-c) as the original phasors.

2. Negative Sequence - three phasors, equal in magnitude, 120^o apart, with the opposite sequence (a-c-b) of the original phasors.

3. Zero Sequence - three identical phasors (i.e. equal in magnitude, with no relative phase displacement).

The original set of phasors is written in terms of the symmetrical components as follows:

$$\tilde{V}_a = \tilde{V}_{a0} + \tilde{V}_{a1} + \tilde{V}_{a2} \ ,$$

$$\tilde{V}_b = \tilde{V}_{b0} + \tilde{V}_{b1} + \tilde{V}_{b2} \ ,$$

$$\tilde{V}_c = \tilde{V}_{c0} + \tilde{V}_{c1} + \tilde{V}_{c2} \ ,$$

where 0 indicates zero sequence, 1 indicates positive sequence, and 2 indicates negative sequence.

The relationship among the sequence components for a-b-c are

Positive Sequence	Negative Sequence	Zero Sequence
$\tilde{V}_{b1} = \tilde{V}_{a1} \bullet 1\angle -120^o$	$\tilde{V}_{b2} = \tilde{V}_{a2} \bullet 1\angle +120^o$	$\tilde{V}_{a0} = \tilde{V}_{b0} = \tilde{V}_{c0}$
$\tilde{V}_{c1} = \tilde{V}_{a1} \bullet 1\angle +120^o$	$\tilde{V}_{c2} = \tilde{V}_{a2} \bullet 1\angle -120^o$	

The symmetrical components of all a-b-c voltages are usually written in terms of the symmetrical components of phase a by defining

$a = 1\angle +120^o$, so that $a^2 = 1\angle +240^o = 1\angle -120^o$, and $a^3 = 1\angle +360^o = 1\angle 0^o$.

Substituting into the previous equations for $\tilde{V}_a, \tilde{V}_b, \tilde{V}_c$ yields

$$\tilde{V}_a = \tilde{V}_{a0} + \tilde{V}_{a1} + \tilde{V}_{a2} \ ,$$

$$\tilde{V}_b = \tilde{V}_{a0} + a^2\tilde{V}_{a1} + a\tilde{V}_{a2} \ ,$$

$$\tilde{V}_c = \tilde{V}_{a0} + a\tilde{V}_{a1} + a^2\tilde{V}_{a2} \ .$$

In matrix form, the above equations become

$$\begin{bmatrix} \tilde{V}_a \\ \tilde{V}_b \\ \tilde{V}_c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \begin{bmatrix} \tilde{V}_{a0} \\ \tilde{V}_{a1} \\ \tilde{V}_{a2} \end{bmatrix} ,$$

or in matrix form

$$\tilde{V}_{abc} = T \bullet \tilde{V}_{012} \ \text{, and } \tilde{V}_{012} = T^{-1} \bullet \tilde{V}_{abc} \ ,$$

where transformation matrix T is

$$T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix} \text{, and } T^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} .$$

If $\tilde{V}_{abc}$ represents a balanced set (i.e. $\tilde{V}_b = \tilde{V}_a \bullet 1\angle -120^{\circ} = a^2\tilde{V}_a$, $\tilde{V}_c = \tilde{V}_a \bullet 1\angle +120^{\circ} = a\tilde{V}_a$), then substituting into $\tilde{V}_{012} = T^{-1} \bullet \tilde{V}_{abc}$ yields

$$\begin{bmatrix} \tilde{V}_{a0} \\ \tilde{V}_{a1} \\ \tilde{V}_{a2} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \begin{bmatrix} \tilde{V}_a \\ a^2\tilde{V}_a \\ a\tilde{V}_a \end{bmatrix} = \begin{bmatrix} 0 \\ \tilde{V}_a \\ 0 \end{bmatrix} .$$

Hence, balanced voltages or currents have only positive sequence components, and the positive sequence components equal the corresponding phase a voltages or currents.

If $\tilde{V}_{abc}$ is an identical set (i.e. $\tilde{V}_a = \tilde{V}_b = \tilde{V}_c$), substituting into $\tilde{V}_{012} = T^{-1} \bullet \tilde{V}_{abc}$ yields

$$\begin{bmatrix} \tilde{V}_{a0} \\ \tilde{V}_{a1} \\ \tilde{V}_{a2} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \begin{bmatrix} \tilde{V}_a \\ \tilde{V}_a \\ \tilde{V}_a \end{bmatrix} = \begin{bmatrix} \tilde{V}_a \\ 0 \\ 0 \end{bmatrix} ,$$

which means that V_a, V_b, V_c have only zero sequence components, and that these components are identical and equal to V_a .

Notice from the top row of $\tilde{V}_{012} = T^{-1} \bullet \tilde{V}_{abc}$ that V_0 is one-third of the sum of the three phase voltages or currents. Therefore, since the sum of three line-to-line voltages is identically zero due to Kirchhoff's voltage law, line-to-line voltages can have no zero sequence components.

2.2 Relationship Between Zero Sequence Currents and Neutral Current

Consider the relationship between zero sequence current and neutral current The zero sequence current is

$$\tilde{I}_{a0} = \frac{1}{3}\left(\tilde{I}_a + \tilde{I}_b + \tilde{I}_c\right) ,$$

and, from Kirchhoff's current law, the neutral current is

$$\tilde{I}_n = -\left(\tilde{I}_a + \tilde{I}_b + \tilde{I}_c\right) .$$

Because the positive and negative sequence components of the a-b-c currents sum to zero, while the zero sequence components are additive, then $\tilde{I}_n = -3\tilde{I}_{a0}$. Therefore, in a four-wire, three-phase system, the neutral current is three-times the zero sequence current. In a three-wire, three-phase system, there is no zero sequence current.

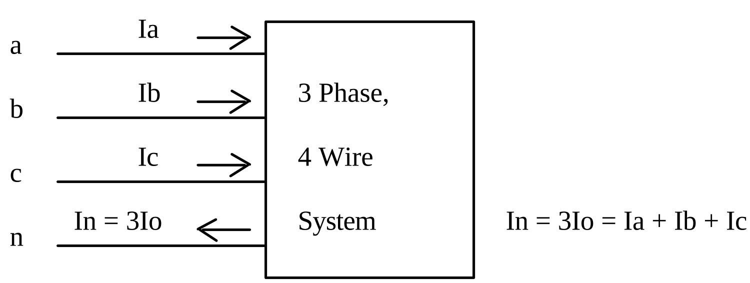

Figure 2.1: Relationship Between Zero Sequence Currents and Neutral Current

2.3 Decoupling in Systems with Balanced Impedances

In a three-phase system with balanced impedances, the relationship among voltage, current, and impedance has the form

$$\begin{bmatrix} \tilde{V}_a \\ \tilde{V}_b \\ \tilde{V}_c \end{bmatrix} = \begin{bmatrix} S & M & M \\ M & S & M \\ M & M & S \end{bmatrix} \begin{bmatrix} \tilde{I}_a \\ \tilde{I}_b \\ \tilde{I}_c \end{bmatrix} , \text{ or } \tilde{V}_{abc} = Z_{abc} \bullet \tilde{I}_{abc} ,$$

where S represents the self impedances of the phases, and M represents the mutual impedances. This equation can be expressed in terms of sequence components by substituting $\tilde{V}_{abc} = T \bullet \tilde{V}_{012}$ and $\tilde{I}_{abc} = T \bullet \tilde{I}_{012}$, yielding

$$T \bullet \tilde{V}_{012} = Z_{abc} \bullet T \bullet \tilde{I}_{012} \ .$$

Premultiplying by T^{-1} yields

$$\tilde{V}_{012} = T^{-1}Z_{abc} \bullet T \bullet \tilde{I}_{012} = Z_{012} \bullet \tilde{I}_{012} \ ,$$

where $Z_{012} = T^{-1}Z_{abc} \bullet T$. The symmetric form of Z_{abc} given above yields

$$Z_{012} = \begin{bmatrix} S+2M & 0 & 0 \\ 0 & S-M & 0 \\ 0 & 0 & S-M \end{bmatrix},$$

which means that when working in sequence components, a circuit with symmetric impedances is decoupled into three separate impedance networks with $Z_0 = S + 2M$, and $Z_1 = Z_2 = S - M$. Furthermore, if the voltages and currents are balanced, then only the positive sequence circuit must be studied.

In summary, symmetrical components are useful when studying either of the following two situations:

1. Symmetric networks with balanced voltages and currents. In that case, only the positive sequence network must be studied, and that network is the "one-line" network.

2. Symmetric networks with unbalanced voltages and currents. In that case, decoupling applies, and three separate networks must be studied (i.e. positive, negative, and zero sequences). The sequence components of the voltages and currents can be transformed back to a-b-c by using the *T* transformation matrix.

2.4 Power

For a three-phase circuit, with voltages referenced to neutral,

$$P_{abc} = \tilde{V}_{an}\tilde{I}_a^* + \tilde{V}_{bn}\tilde{I}_b^* + \tilde{V}_{cn}\tilde{I}_c^* = \tilde{V}_{abc}^T \bullet \tilde{I}_{abc}^* \ .$$

Substituting in $\tilde{V}_{abc} = T \bullet \tilde{V}_{012}$ and $\tilde{I}_{abc} = T \bullet \tilde{I}_{012}$ yields

$$P_{abc} = \tilde{V}_{012}^T \bullet T^T \bullet T^* \bullet \tilde{I}_{012}^* \ .$$

Since $T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix}$, then $T^T = T$. Also, $T^* = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix}$. Therefore,

$$T^T \bullet T^* = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \text{ so that } P_{abc} = \tilde{V}_{012}^T \bullet \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \bullet \tilde{I}_{012}^* \text{ , or}$$

$$P_{abc} = 3\left(\tilde{V}_{a0}\tilde{I}_{a0}^* + \tilde{V}_{a1}\tilde{I}_{a1}^* + \tilde{V}_{a2}\tilde{I}_{a2}^*\right) \ .$$

Note the factor of three. If desired, the following power invariant transformation can be used to avoid the factor of three:

$$T = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{bmatrix}, \ T^{-1} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix},$$

so that $P_{abc} = \tilde{V}_{a0}\tilde{I}_{a0}^* + \tilde{V}_{a1}\tilde{I}_{a1}^* + \tilde{V}_{a2}\tilde{I}_{a2}^*$.