Section 3. Transformers, Transmission Lines, and Underground Cables

Transformers. Transformer phase shift. Wye-delta connections and impact on zero sequence. Inductance and capacitance calculations for transmission lines. GMR, GMD, L, and C matrices, effect of ground conductivity. Underground cables.

3.1 Transformers

3.1.1 Equivalent Circuits

The standard transformer equivalent circuit used in power system simulation is shown below, where the R and X terms represent the series resistance and leakage reactance, and N1 and N2 represent the transformer turns. Note that the shunt terms are usually ignored in the model..

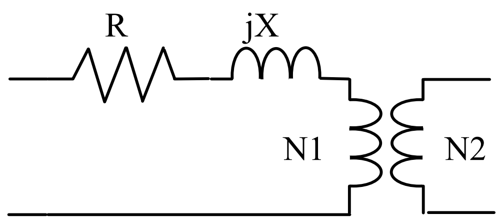


Figure 3.1: Power System Model for Transformer

Three-phase transformers can consist of either three separate single-phase transformers, or three windings on a three-legged, four-legged, or five-legged core. The high-voltage and low-voltage sides can be connected independently in either wye or delta.

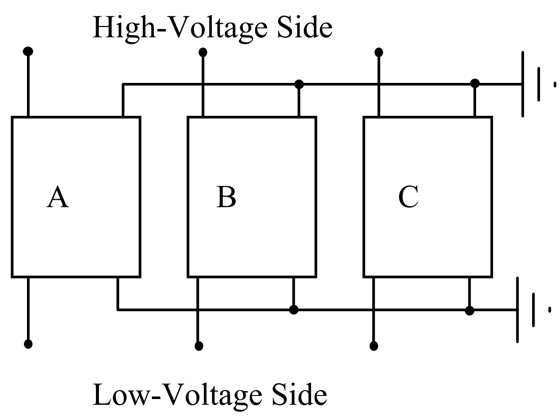


Figure 3.2: A Three-Phase Ground-Wye Grounded-Wye Transformer

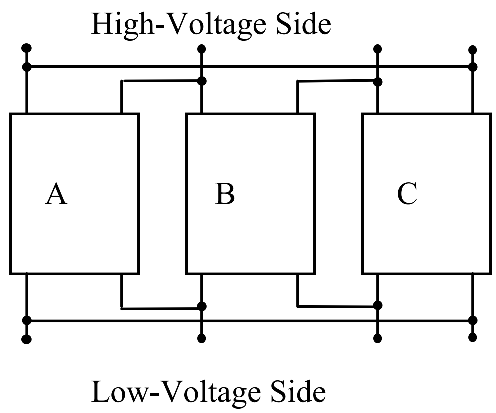


Figure 3.3: A Three-Phase Delta-Delta Transformer

The transformer impedances consist of winding resistances and leakage reactances. There are no mutual resistances, and the mutual leakage reactances between the separate phase a-b-c coils are negligible. Hence, in symmetrical components, $S = R + jX$, and $M = 0$, so that $S + 2M = S - M = R + jX$, so therefore the positive and negative sequence impedances of a transformer are

$$Z_1 = Z_2 = R + jX \quad .$$

One must remember that no zero sequence currents can flow into a three-wire connection. Therefore, the zero sequence impedance of a transformer depends on the winding connections. In the case where one side of a transformer is connected grounded-wye, and the other side is delta, circulating zero sequence currents can be induced in the delta winding. In that case, the zero sequence impedance "looking into" the transformer is different on the two sides.

The zero sequence equivalent circuits for three-phase transformers is given in Figure 3.4.

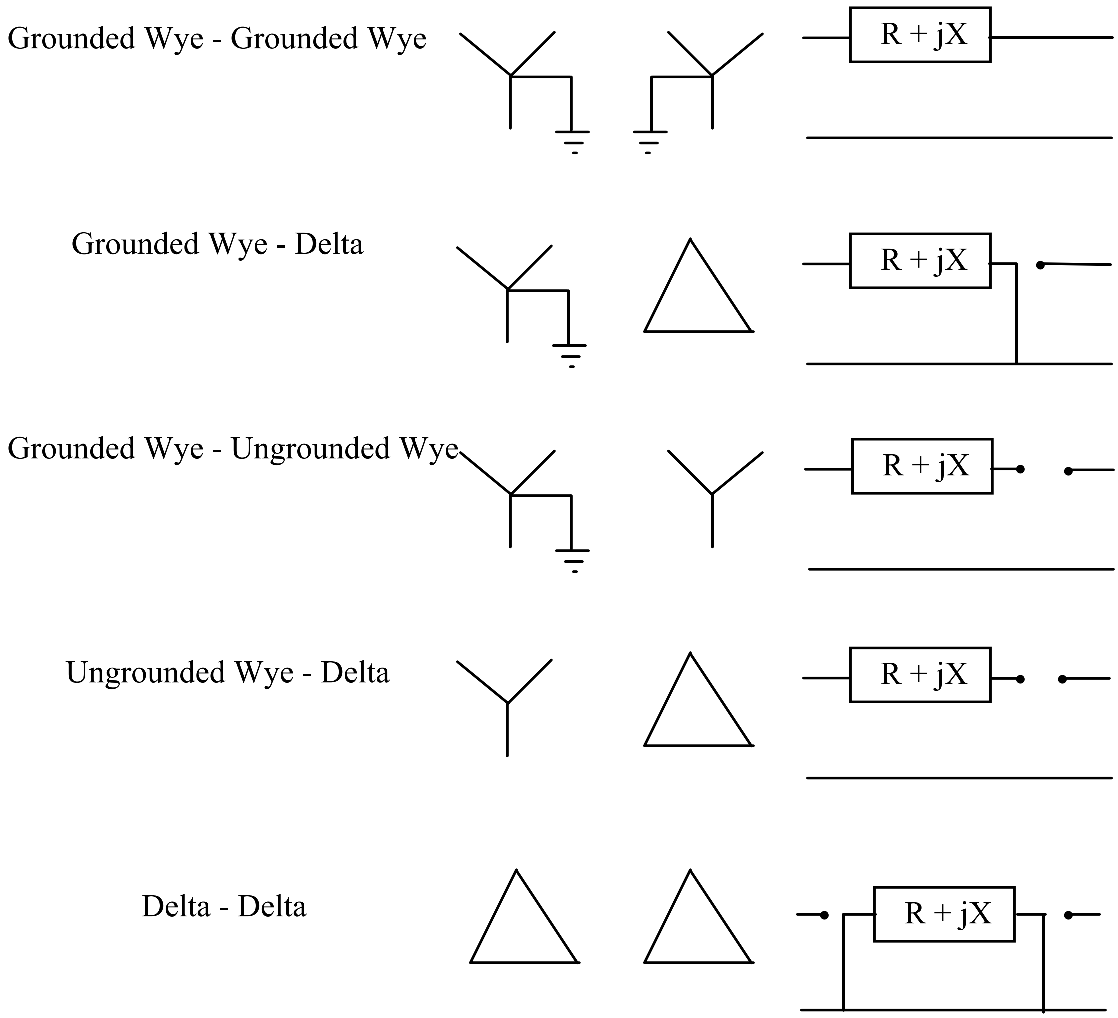


Figure 3.4: Zero Sequence Impedance Equivalent Circuits for Three-Phase Transformers

A wye-delta transformer connection introduces a 30^{o} phase shift in positive/negative sequence voltages and currents because of the relative shift between line-to-neutral and line-to-ground voltages. Transformers are labeled so that

1. High side positive sequence voltages and currents lead those on the low side by 30^{o}.

2. High side negative sequence voltages and currents lag those on the low side by 30^{o}.

3. There is no phase shift for zero sequence.

Transformer tap magnitudes can be adjusted to control voltage, and transformer phase shifts can be adjusted to control active power flow. The effect of these "off-nominal" adjustments can be incorporated into a pi-equivalent circuit model for a transformer.

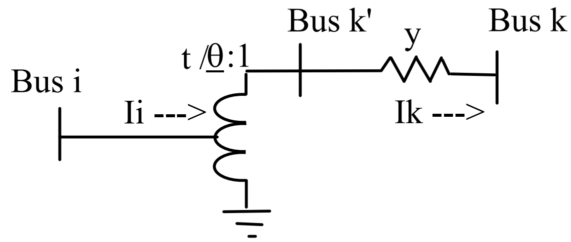


Figure 3.5: Off-Nominal Transformer Circuit Model

Assume that the transformer in Figure 3.5 has complex "off-nominal" tap $t\angle\theta_t$ t and series admittance y. The relationship between the voltage on opposite sides of the transformer tap is $\tilde{V}_{k'} = \dfrac{\tilde{V}_i}{t\angle\theta_t}$, and since the power on both sides of the ideal transformer must be the same, then $\tilde{V}_i\tilde{I}_i^* = \tilde{V}_{k'}\tilde{I}_{k'}^*$, implying that $\tilde{I}_{k'} = \tilde{I}_i t\angle -\theta_t$. Now, suppose that the transformer can be modeled by the following pi-equivalent circuit of Figure 3.6:

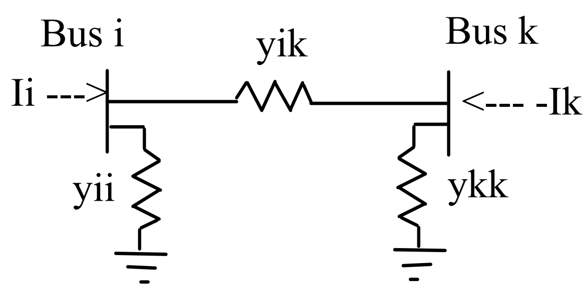


Figure 3.6: Pi-Equivalent Model of Transformer

Admittances y_{ii}, y_{ik}, and y_{kk} can be found so that the above circuit is equivalent to Figure 3.5. This can be accomplished by forcing the terminal behavior to be the same. For the above circuit, the appropriate equations are

$$\tilde{I}_i = \left(\tilde{V}_i - \tilde{V}_k\right)y_{ik} + \tilde{V}_i y_{ii} \text{ , and } -\tilde{I}_k = \left(\tilde{V}_k - \tilde{V}_i\right)y_{ik} + \tilde{V}_k y_{kk} \ ,$$

or in matrix form

$$\begin{bmatrix} \tilde{I}_i \\ -\tilde{I}_k \end{bmatrix} = \begin{bmatrix} y_{ii} + y_{ik} & -y_{ik} \\ -y_{ik} & y_{kk} + y_{ik} \end{bmatrix} \begin{bmatrix} \tilde{V}_i \\ \tilde{V}_k \end{bmatrix} .$$

For Figure 3.5, the terminal equations are

$$\tilde{I}_k = \left(\tilde{V}_{k'} - \tilde{V}_k\right)y = \left(\frac{\tilde{V}_i}{t\angle\theta_t} - \tilde{V}_k\right)y \ ,$$

and since

$$\tilde{I}_i = \frac{\tilde{I}_k}{t\angle -\theta_t} ,$$

then

$$\tilde{I}_i = \left(\frac{\tilde{V}_i}{t\angle\theta_t \bullet t\angle -\theta_t} - \frac{\tilde{V}_k}{t\angle -\theta_t} \right) y .$$

In matrix form,

$$\begin{bmatrix} \tilde{I}_i \\ -\tilde{I}_k \end{bmatrix} = \begin{bmatrix} \dfrac{y}{t\angle\theta_t \bullet t\angle -\theta_t} & \dfrac{-y}{t\angle -\theta_t} \\ \dfrac{-y}{t\angle\theta_t} & y \end{bmatrix} \begin{bmatrix} \tilde{V}_i \\ \tilde{V}_k \end{bmatrix} .$$

Comparing the two sets of terminal equations shows that equality can be reached if the shunt branch in the equivalent circuit, y_{ik}, can have two values:

$y_{ik} = \dfrac{y}{t\angle -\theta_t}$ from the perspective of Kirchhoff's current law at bus i, $\quad y_{ik} = \dfrac{y}{t\angle\theta_t}$ from the perspective of Kirchhoff's current law at bus k.

Note that if the tap does not include an off-nominal phase shift, then $y_{ik} = \dfrac{y}{t}$ from either direction.

Next, solving for y_{ii} and y_{kk} yields

$$y_{ii} = \frac{y}{t\angle\theta_t \bullet t\angle -\theta_t} - \frac{y}{t\angle -\theta_t} = \frac{y}{t\angle -\theta_t}\left(\frac{1}{t\angle\theta_t} - 1 \right) ,$$

$$y_{kk} = y - \frac{y}{t\angle\theta_t} = y\left(1 - \frac{1}{t\angle\theta_t} \right) .$$

3.1.2 Neutral Grounding Impedance

If the wye-side of a transformer or wye-connected load is grounded through a grounding impedance Z_g, the grounding impedance is "invisible" to the positive and negative sequence currents since their corresponding voltages at the wye-point is always zero due to symmetry. However, since the neutral current is three-times the zero sequence current, the voltage drop on the grounding impedance is $3I_{ao}$. For that reason, the zero sequence equivalent circuit for a grounding impedance must contain $3Z_g$.

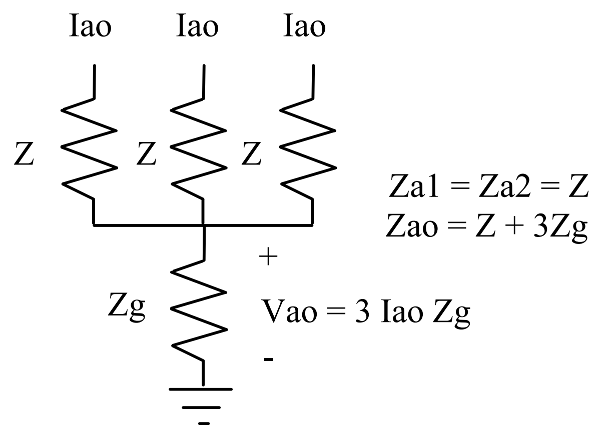
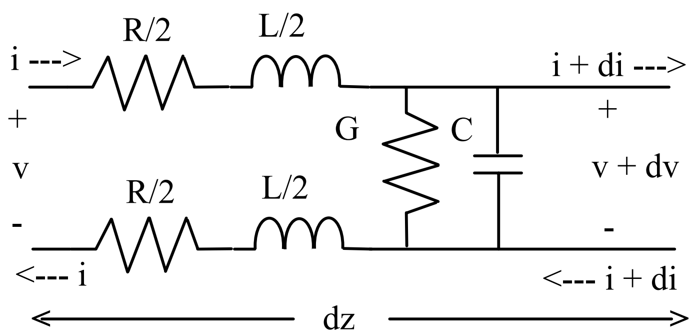


Figure 3.7: Effect of Grounding Impedance on Sequence Impedances

3.2 Transmission Lines

3.2.1 Equivalent Circuit

The power system model for transmission lines is developed from the conventional distributed parameter model, shown in Figure 3.8.



R, L, G, C per unit length

Figure 3.8: Distributed Parameter Model for Transmission Line

Once the values for distributed parameters resistance R, inductance L, conductance G, and capacitance are known (units given in per unit length), then either "long line" or "short line" models can be used, depending on the electrical length of the line.

Assuming for the moment that R, L, G, and C are known, the relationship between voltage and current on the line may be determined by writing Kirchhoff's voltage law (KVL) around the outer loop in Figure 3.8, and by writing Kirchhoff's current law (KCL) at the right-hand node.

KVL yields

$$-v + \frac{Rdz}{2} i + \frac{Ldz}{2} \frac{\partial i}{\partial t} + v + dv + \frac{Rdz}{2} i + \frac{Ldz}{2} \frac{\partial i}{\partial t} = 0 .$$

This yields the change in voltage per unit length, or

$$\frac{\partial v}{\partial z} = -Ri - L\frac{\partial i}{\partial t} \ ,$$

which in phasor form is

$$\frac{\partial \tilde{V}}{\partial z} = -\left(R + j\omega L\right)\tilde{I} \ .$$

KCL at the right-hand node yields

$$-i + i + di + Gdz\left(v + dv\right) + Cdz\frac{\partial\left(v + dv\right)}{\partial t} = 0 \ .$$

If dv is small, then the above formula can be approximated as

$di = -\left(Gdz\right)v - Cdz\dfrac{\partial v}{\partial t}$, or $\dfrac{\partial i}{\partial z} = -Gv - C\dfrac{\partial v}{\partial t}$, which in phasor form is

$$\frac{\partial \tilde{I}}{\partial z} = -\left(G + j\omega C\right)\tilde{V} \ .$$

Taking the partial derivative of the voltage phasor equation with respect to z yields

$$\frac{\partial^2 \tilde{V}}{\partial z^2} = -\left(R + j\omega L\right)\frac{\partial \tilde{I}}{\partial z} \ .$$

Combining the two above equations yields

$\dfrac{\partial^2 \tilde{V}}{\partial z^2} = \left(R + j\omega L\right)\left(G + j\omega C\right)\tilde{V} = \gamma^2\tilde{V}$, where $\gamma = \sqrt{\left(R + j\omega L\right)\left(G + j\omega C\right)} = \alpha + j\beta$, and where γ, α, and β are the propagation, attenuation, and phase constants, respectively.

The solution for $\tilde{V}$ is

$$\tilde{V}(z) = Ae^{\gamma z} + Be^{-\gamma z} \ .$$

A similar procedure for solving $\tilde{I}$ yields

$$\tilde{I}(z) = \frac{-Ae^{\gamma z} + Be^{-\gamma z}}{Z_o} \ ,$$

where the characteristic or "surge" impedance Z_o is defined as

$$Z_o = \sqrt{\frac{(R + j\omega L)}{(G + j\omega C)}} \ .$$

Constants A and B must be found from the boundary conditions of the problem. This is usually accomplished by considering the terminal conditions of a transmission line segment that is d meters long, as shown in Figure 3.9.

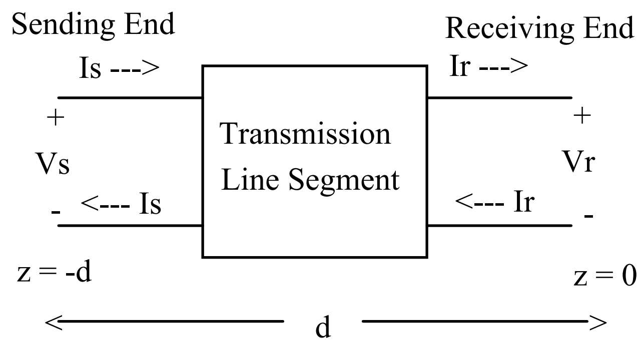


Figure 3.9: Transmission Line Segment

In order to solve for constants A and B, the voltage and current on the receiving end is assumed to be known so that a relation between the voltages and currents on both sending and receiving ends may be developed.

Substituting $z = 0$ into the equations for the voltage and current (at the receiving end) yields

$$\widetilde{V}_R = A + B, \widetilde{I}_R = \frac{-(A - B)}{Z_o} \ .$$

Solving for A and B yields

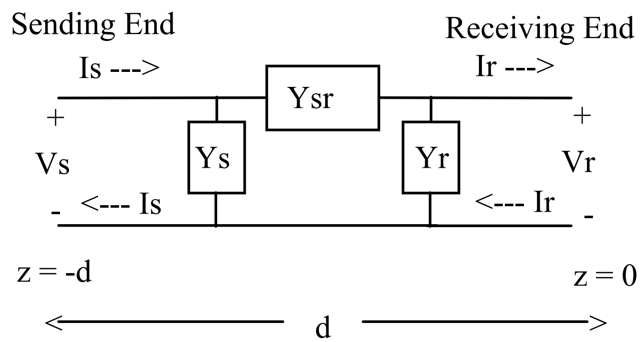
$$A = \frac{\widetilde{V}_R - Z_o I_R}{2}, B = \frac{\widetilde{V}_R + Z_o I_R}{2} \ .$$

Substituting into the $\widetilde{V}(z)$ and $\widetilde{I}(z)$ equations yields

$$\widetilde{V}_S = \widetilde{V}_R \cosh(\gamma d) + Z_0 \widetilde{I}_R \sinh(\gamma d) \ ,$$

$$\widetilde{I}_S = \frac{\widetilde{V}_R}{Z_o} \sinh(\gamma d) + \widetilde{I}_R \cosh(\gamma d) \ .$$

A pi equivalent model for the transmission line segment can now be found, in a similar manner as it was for the off-nominal transformer. The results are given in Figure 3.10.



$$Y_S = Y_R = \frac{\tanh\left(\frac{\gamma d}{2}\right)}{Z_o} \ , \ Y_{SR} = \frac{1}{Z_o \sinh(\gamma d)} \ , \ Z_o = \sqrt{\frac{(R + j\omega L)}{(G + j\omega C)}} \ , \ \gamma = \sqrt{(R + j\omega L)(G + j\omega C)}$$

R, *L*, *G*, *C* per unit length

Figure 3.10: Pi Equivalent Circuit Model for Distributed Parameter Transmission Line

Shunt conductance G is usually neglected in overhead lines, but it is not negligible in underground cables.

For electrically "short" overhead transmission lines, the hyperbolic pi equivalent model simplifies to a familiar form. Electrically short implies that d < 0.05λ, where wavelength $\lambda = \frac{3\left(10^8\right) m/s}{f\sqrt{\varepsilon_r}\, Hz}$ = 5000 kM @ 60 Hz, or 6000 kM @ 50 Hz. Therefore, electrically short overhead lines have d < 250 kM @ 60 Hz, and d < 300 kM @ 50 Hz. For underground cables, the corresponding distances are less since cables have somewhat higher relative permittivities (i.e. $\varepsilon_r \approx 2.5$).

Substituting small values of γd into the hyperbolic equations, and assuming that the line losses are negligible so that $G = R = 0$, yields

$$Y_S = Y_R = \frac{j\omega C d}{2} \ , \text{and } \ Y_{SR} = \frac{1}{j\omega L d} \ .$$

Then, including the series resistance yields the conventional "short" line model shown in Figure 3.11, where half of the capacitance of the line is lumped on each end.

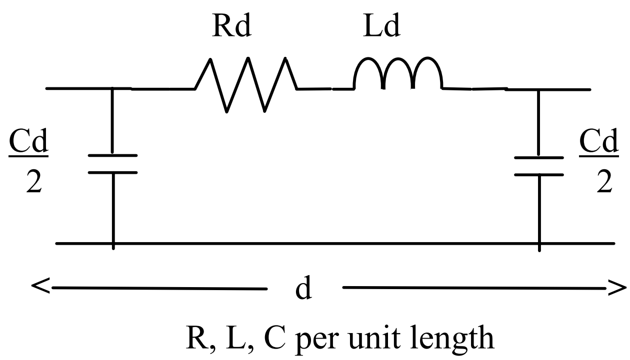


Figure 3.11: Standard Short Line Pi Equivalent Model for a Transmission Line

3.2.2 Capacitance

Overhead transmission lines consist of wires that are parallel to the surface of the earth. To determine the capacitance of a transmission line, first consider the capacitance of a single wire over the earth. Wires over the earth are typically modeled as line charges ρ_l Coulombs per meter of length, and the relationship between the applied voltage and the line charge is the capacitance.

A line charge in space has a radially outward electric field described as

$$\overline{E} = \frac{q_l}{2\pi\varepsilon_o r}\hat{a}_r \text{ Volts per meter .}$$

This electric field causes a voltage drop between two points at distances $r = a$ and $r = b$ away from the line charge. The voltage is found by integrating electric field, or

$$V_{ab} = \int_{r=a}^{r=b} \overline{E} \bullet r\hat{a}_r = \frac{q_l}{2\pi\varepsilon_o} \ln\left(\frac{b}{a}\right) \text{ V.}$$

If the wire is above the earth, it is customary to treat the earth's surface as a perfect conducting plane, which can be modeled as an equivalent image line charge $-q_l$ lying at an equal distance below the surface, as shown in Figure 3.12.

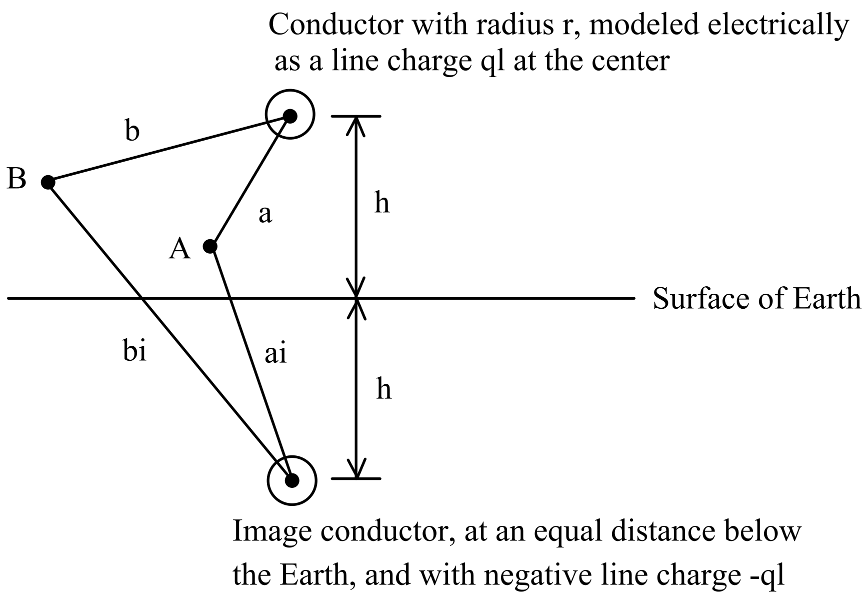


Figure 3.12: Line Charge q_l at Center of Conductor Located h Meters Above the Earth

From superposition, the voltage difference between points A and B is

$$V_{ab} = \int_{r=a}^{r=b} \overline{E}_{\rho} \bullet \hat{a}_r + \int_{r=ai}^{r=bi} \overline{E}_{\rho i} \bullet \hat{a}_r = \frac{q_l}{2\pi\varepsilon_o}\left[\ln\left(\frac{b}{a}\right) - \ln\left(\frac{bi}{ai}\right)\right] = \frac{q_l}{2\pi\varepsilon_o}\ln\left(\frac{b \bullet ai}{a \bullet bi}\right) .$$

If point B lies on the earth's surface, then from symmetry, $b = bi$, and the voltage of point A with respect to ground becomes

$$V_{ag} = \frac{q_l}{2\pi\varepsilon_o}\ln\left(\frac{ai}{a}\right) .$$

The voltage at the surface of the wire determines the wire's capacitance. This voltage is found by moving point A to the wire's surface, corresponding to setting $a = r$, so that

$$V_{rg} \approx \frac{q_l}{2\pi\varepsilon_o}\ln\left(\frac{2h}{r}\right) \text{ for } h >> r.$$

The exact expression, which accounts for the fact that the equivalent line charge drops slightly below the center of the wire, but still remains within the wire, is

$$V_{rg} = \frac{q_l}{2\pi\varepsilon_o}\ln\left(\frac{h + \sqrt{h^2 + r^2}}{r}\right) .$$

The capacitance of the wire is defined as $C = \dfrac{\rho_l}{V_{rg}}$ which, using the approximate voltage formula above, becomes

$$C = \frac{2\pi\varepsilon_o}{\ln\left(\dfrac{2h}{r}\right)} \text{ Farads per meter of length.}$$

When several conductors are present, then the capacitance of the configuration must be given in matrix form. Consider phase a-b-c wires above the earth, as shown in Figure 3.13.

Three Conductors Represented by Their Equivalent Line Charges

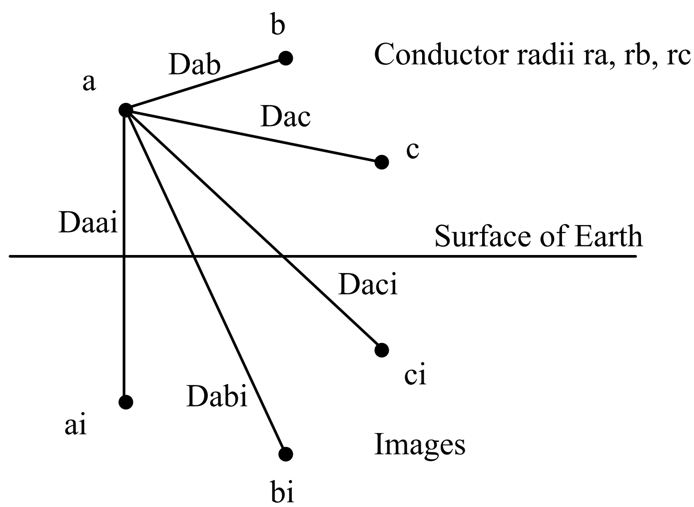


Figure 3.13: Three Conductors Above the Earth

Superposing the contributions from all three line charges and their images, the voltage at the surface of conductor a is given by

$$V_{ag} = \frac{1}{2\pi\varepsilon_o}\left[q_a \ln\frac{D_{aai}}{r_a} + q_b \ln\frac{D_{abi}}{D_{ab}} + q_c \ln\frac{D_{aci}}{D_{ac}}\right].$$

The voltages for all three conductors can be written in generalized matrix form as

$$\begin{bmatrix} V_{ag} \\ V_{bg} \\ V_{cg} \end{bmatrix} = \frac{1}{2\pi\varepsilon_o}\begin{bmatrix} p_{aa} & p_{ab} & p_{ac} \\ p_{ba} & p_{bb} & p_{bc} \\ p_{ca} & p_{cb} & p_{cc} \end{bmatrix}\begin{bmatrix} q_a \\ q_b \\ q_c \end{bmatrix}, \text{ or } V_{abc} = \frac{1}{2\pi\varepsilon_o} P_{abc} Q_{abc} ,$$

where

$$p_{aa} = \ln \frac{D_{aai}}{r_a} \ , \ p_{ab} = \ln \frac{D_{abi}}{D_{ab}} \ \text{, etc., and}$$

r_a is the radius of conductor a,

D_{aai} is the distance from conductor a to its own image (i.e. twice the height of conductor a above ground),

D_{ab} is the distance from conductor a to conductor b,

$D_{abi} = D_{bai}$ is the distance between conductor a and the image of conductor b (which is the same as the distance between conductor b and the image of conductor a), etc.

A Matrix Approach for Finding C

From the definition of capacitance, $Q = CV$, then the capacitance matrix can be obtained via inversion, or

$$C_{abc} = 2\pi\varepsilon_o P_{abc}^{-1} \ .$$

If ground wires are present, the dimension of the problem increases proportionally. For example, in a three-phase system with two ground wires, the dimension of the *P* matrix is 5 x 5. However, given the fact that the line-to-ground voltage of the ground wires is zero, equivalent 3 x 3 *P* and *C* matrices can be found by using matrix partitioning and a process known as Kron reduction. First, write the $V = PQ$ equation as follows:

$$\begin{bmatrix} V_{ag} \\ V_{bg} \\ V_{cg} \\ - \\ V_{vg} = 0 \\ V_{wg} = 0 \end{bmatrix} = \frac{1}{2\pi\varepsilon_o} \begin{bmatrix} P_{abc}(3\text{x}3) & | & P_{abc,vw}(3\text{x}2) \\ - & & - \\ P_{vw,abc}(2\text{x}3) & | & P_{vw}(2\text{x}2) \end{bmatrix} \begin{bmatrix} q_a \\ q_b \\ q_c \\ - \\ q_v \\ q_w \end{bmatrix},$$

or

$$\begin{bmatrix} V_{abc} \\ V_{vw} \end{bmatrix} = \frac{1}{2\pi\varepsilon_o} \begin{bmatrix} P_{abc} & P_{abc,vw} \\ P_{vw,abc} & P_{vw} \end{bmatrix} \begin{bmatrix} Q_{abc} \\ Q_{vw} \end{bmatrix},$$

where subscripts *v* and *w* refer to ground wires w and v, and where the individual *P* matrices are formed as before. Since the ground wires have zero potential, then

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \frac{1}{2\pi\varepsilon_o}\left[P_{vw,abc}Q_{abc} + P_{vw}Q_{vw}\right],$$

so that

$$Q_{vw} = -P_{vw}^{-1}\left[P_{vw,abc}Q_{abc}\right] .$$

Substituting into the V_{abc} equation above, and combining terms, yields

$$V_{abc} = \frac{1}{2\pi\varepsilon_o}\left[P_{abc}Q_{abc} - P_{abc,vw}P_{vw}^{-1}P_{vw,abc}Q_{abc}\right] = \frac{1}{2\pi\varepsilon_o}\left[P_{abc} - P_{abc,vw}P_{vw}^{-1}P_{vw,abc}\right]Q_{abc} ,$$

or

$$V_{abc} = \frac{1}{2\pi\varepsilon_o}\left[P_{abc}^{'}\right]Q_{abc} \text{ , so that}$$

$$Q_{abc} = C_{abc}^{'}V_{abc} \text{ , where } C_{abc}^{'} = 2\pi\varepsilon_o\left[P_{abc}^{'}\right]^{-1} .$$

Therefore, the effect of the ground wires can be included into a 3 x 3 equivalent capacitance matrix.

An alternative way to find the equivalent 3 x 3 capacitance matrix $C_{abc}^{'}$ is to

- obtain the 5 x 5 *C* matrix by inverting the 5 x 5 *P*, and then
- Kron reduce the 5 x 5 *C* directly.

Computing 012 Capacitances from Matrices

Once the 3 x 3 $C_{abc}^{'}$ matrix is found by either of the above two methods, 012 capacitances can be determined by averaging the diagonal terms, and averaging the off-diagonal terms of, $C_{abc}^{'}$ to produce

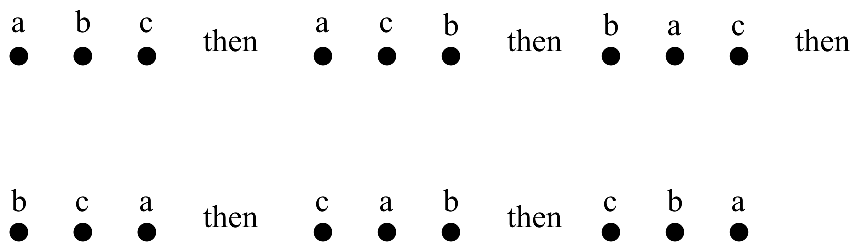
$$C_{abc}^{avg} = \begin{bmatrix} C_S & C_M & C_M \\ C_M & C_S & C_S \\ C_M & C_M & C_S \end{bmatrix} .$$

C_{abc}^{avg} has the special symmetric form for diagonalization into 012 components, which yields

$$C_{012}^{avg} = \begin{bmatrix} C_S + 2C_M & 0 & 0 \\ 0 & C_S - C_M & 0 \\ 0 & 0 & C_S - C_M \end{bmatrix}.$$

The Approximate Formulas for 012 Capacitances

Asymmetries in transmission lines prevent the *P* and *C* matrices from having the special form that allows their diagonalization into decoupled positive, negative, and zero sequence impedances. Transposition of conductors can be used to nearly achieve the special symmetric form and, hence, improve the level of decoupling. Conductors are transposed so that each one occupies each phase position for one-third of the lines total distance. An example is given below in Figure 3.14, where the radii of all three phases are assumed to be identical.



where each configuration occupies one-sixth of the total distance

Figure 3.14: Transposition of A-B-C Phase Conductors

For this mode of construction, the average *P* matrix (or Kron reduced *P* matrix if ground wires are present) has the following form:

$$P_{abc}^{avg} = \frac{1}{6}\begin{bmatrix} p_{aa} & p_{ab} & p_{ac} \\ \bullet & p_{bb} & p_{bc} \\ \bullet & \bullet & p_{cc} \end{bmatrix} + \frac{1}{6}\begin{bmatrix} p_{aa} & p_{ac} & p_{ab} \\ \bullet & p_{cc} & p_{bc} \\ \bullet & \bullet & p_{bb} \end{bmatrix} + \frac{1}{6}\begin{bmatrix} p_{bb} & p_{ab} & p_{bc} \\ \bullet & p_{aa} & p_{ac} \\ \bullet & \bullet & p_{cc} \end{bmatrix} +$$

$$\begin{bmatrix} p_{cc} & p_{ac} & p_{bc} \\ \bullet & p_{aa} & p_{ab} \\ \bullet & \bullet & p_{bb} \end{bmatrix} + \frac{1}{6}\begin{bmatrix} p_{bb} & p_{bc} & p_{ab} \\ \bullet & p_{cc} & p_{ac} \\ \bullet & \bullet & p_{aa} \end{bmatrix} + \frac{1}{6}\begin{bmatrix} p_{cc} & p_{bc} & p_{ac} \\ \bullet & p_{bb} & p_{ab} \\ \bullet & \bullet & p_{aa} \end{bmatrix},$$

where the individual *p* terms are described previously. Note that these individual *P* matrices are symmetric, since $D_{ab} = D_{ba}, p_{ab} = p_{ba}$, etc. Since the sum of natural logarithms is the same as the logarithm of the product, P becomes

$$P_{abc}^{avg} = \begin{bmatrix} p_S & p_M & p_M \\ p_M & p_S & p_M \\ p_M & p_M & p_S \end{bmatrix},$$

where

$$p_s = \frac{P_{aa} + P_{bb} + P_{cc}}{3} = \ln \frac{\sqrt[3]{D_{aai} D_{bbi} D_{cci}}}{\sqrt[3]{r_a r_b r_c}} ,$$

and

$$p_M = \frac{P_{ab} + P_{ac} + P_{bc}}{3} = \ln \frac{\sqrt[3]{D_{abi} D_{aci} D_{bci}}}{\sqrt[3]{D_{ab} D_{ac} D_{bc}}} .$$

Since P_{abc}^{avg} has the special property for diagonalization in symmetrical components, then transforming it yields

$$P_{012}^{avg} = \begin{bmatrix} p_0 & 0 & 0 \\ 0 & p_1 & 0 \\ 0 & 0 & p_2 \end{bmatrix} = \begin{bmatrix} p_S + 2p_M & 0 & 0 \\ 0 & p_S - p_M & 0 \\ 0 & 0 & p_S - p_M \end{bmatrix},$$

where

$$p_s - p_M = \ln \frac{\sqrt[3]{D_{aai} D_{bbi} D_{cci}}}{\sqrt[3]{r_a r_b r_c}} - \ln \frac{\sqrt[3]{D_{abi} D_{aci} D_{bci}}}{\sqrt[3]{D_{ab} D_{ac} D_{bc}}} = \ln \frac{\sqrt[3]{D_{aai} D_{bbi} D_{cci}} \sqrt[3]{D_{ab} D_{ac} D_{bc}}}{\sqrt[3]{r_a r_b r_c} \sqrt[3]{D_{abi} D_{aci} D_{bci}}} .$$

Inverting P_{012}^{avg} and multiplying by $2\pi\varepsilon_o$ yields the corresponding 012 capacitance matrix

$$C_{012}^{avg} = \begin{bmatrix} C_0 & 0 & 0 \\ 0 & C_1 & 0 \\ 0 & 0 & C_2 \end{bmatrix} = 2\pi\varepsilon_o \begin{bmatrix} \frac{1}{p_0} & 0 & 0 \\ 0 & \frac{1}{p_1} & 0 \\ 0 & 0 & \frac{1}{p_2} \end{bmatrix} = 2\pi\varepsilon_o \begin{bmatrix} \frac{1}{p_S + 2p_M} & 0 & 0 \\ 0 & \frac{1}{p_S - p_M} & 0 \\ 0 & 0 & \frac{1}{p_S - p_M} \end{bmatrix} .$$

When the a-b-c conductors are closer to each other than they are to the ground, then

$$D_{aai} D_{bbi} D_{cci} \approx D_{abi} D_{aci} D_{bci} ,$$

yielding the conventional approximation

$$p_1 = p_2 = p_S - p_M = \ln \frac{\sqrt[3]{D_{ab} D_{ac} D_{bc}}}{\sqrt[3]{r_a r_b r_c}} = \ln \frac{GMD_{1,2}}{GMR_{1,2}} ,$$

where $GMD_{1,2}$ and $GMR_{1,2}$ are the geometric mean distance (between conductors) and geometric mean radius, respectively, for both positive and negative sequences. Therefore, the positive and negative sequence capacitances become

$$C_1 = C_2 = \frac{2\pi\varepsilon_o}{p_S - p_M} = \frac{2\pi\varepsilon_o}{\ln\frac{GMD_{1,2}}{GMR_{1,2}}} \text{ Farads per meter.}$$

For the zero sequence term,

$$p_0 = p_s + 2p_M = \ln\frac{\sqrt[3]{D_{aai}D_{bbi}D_{cci}}}{\sqrt[3]{r_a r_b r_c}} + 2\ln\frac{\sqrt[3]{D_{abi}D_{aci}D_{bci}}}{\sqrt[3]{D_{ab}D_{ac}D_{bc}}} =$$

$$\ln\sqrt[3]{\frac{(D_{aai}D_{bbi}D_{cci})(D_{abi}D_{aci}D_{bci})^2}{(r_a r_b r)(D_{ab}D_{ac}D_{bc})^2}} \; .$$

Expanding yields

$$p_0 = 3\ln\sqrt[9]{\frac{(D_{aai}D_{bbi}D_{cci})(D_{abi}D_{aci}D_{bci})^2}{(r_a r_b r)(D_{ab}D_{ac}D_{bc})^2}} =$$

$$3\ln\sqrt[9]{\frac{(D_{aai}D_{bbi}D_{cci})(D_{abi}D_{aci}D_{bci})(D_{bai}D_{cai}D_{cbi})}{(r_a r_b r)(D_{ab}D_{ac}D_{bc})(D_{ba}D_{ca}D_{cb})}} \; ,$$

or

$$p_0 = 3\ln\frac{GMD_0}{GMR_0} \; ,$$

where

$$GMD_0 = \sqrt[9]{\left(D_{aai}D_{bbi}D_{cci}\right)\left(D_{abi}D_{aci}D_{bci}\right)\left(D_{bai}D_{cai}D_{cbi}\right)} \; ,$$

$$GMR_0 = \sqrt[9]{\left(r_a r_b r_c\right)\left(D_{ab}D_{ac}D_{bc}\right)\left(D_{ba}D_{ca}D_{cb}\right)} \; .$$

The zero sequence capacitance then becomes

$$C_0 = \frac{2\pi\varepsilon_o}{p_S + 2p_M} = \frac{1}{3}\frac{2\pi\varepsilon_o}{\ln\frac{GMD_0}{GMR_o}} \text{ Farads per meter,}$$

which is one-third that of the entire a-b-c bundle by because it represents the average contribution of only one phase.

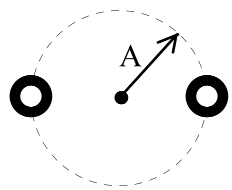
Bundled Phase Conductors

If each phase consists of a symmetric bundle of N identical individual conductors, an equivalent radius can be computed by assuming that the total line charge on the phase divides equally among the N individual conductors. The equivalent radius is

$$r_{eq} = \left[NrA^{N-1} \right]^{\frac{1}{N}} ,$$

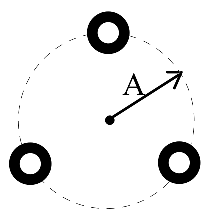
where r is the radius of the individual conductors, and A is the bundle radius of the symmetric set of conductors. Three common examples are shown below in Figure 3.15.

Double Bundle, Each Conductor Has Radius r



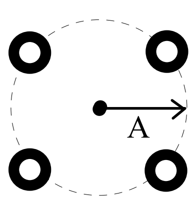
$$r_{eq} = \sqrt{2rA}$$

Triple Bundle, Each Conductor Has Radius r



$$r_{eq} = \sqrt[3]{3rA^2}$$

Quadruple Bundle, Each Conductor Has Radius r



$$r_{eq} = \sqrt[4]{4rA^3}$$

Figure 3.15: Equivalent Radius for Three Common Types of Bundled Phase Conductors

3.2.3 Inductance

The magnetic field intensity produced by a long, straight current carrying conductor is given by Ampere's Circuital Law to be

$$H_\phi = \frac{I}{2\pi r} \text{ Amperes per meter,}$$

where the direction of $\overline{H}$ is given by the right-hand rule.

Magnetic flux density is related to magnetic field intensity by permeability m as follows:

$$\overline{B} = \mu \overline{H} \text{ Webers per square meter,}$$

and the amount of magnetic flux passing through a surface is

$$\Phi = \int \overline{B} \bullet d\overline{s} \text{ Webers,}$$

where the permeability of free space is $\mu_o = 4\pi\left(10^{-7}\right)$.

Two Parallel Wires in Space

Now, consider a two-wire circuit that carries current I, as shown in Figure 3.16.

Two current-carying wires with radii r

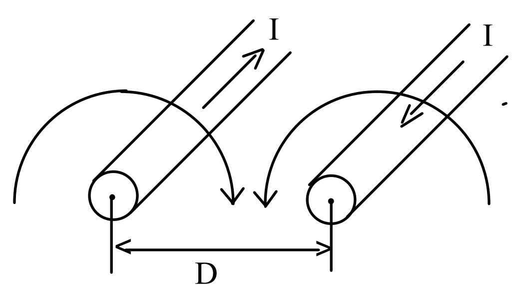


Figure 3.16: A Circuit Formed by Two Long Parallel Conductors

The amount of flux linking the circuit (i.e. passes between the two wires) is found to be

$$\Phi = \int_r^{D-r} \frac{\mu_o I}{2\pi x} dx + \int_r^{D-r} \frac{\mu_o I}{2\pi x} dx = \frac{\mu_o I}{\pi} \ln \frac{D-r}{r} \text{ Henrys per meter length.}$$

From the definition of inductance,

$$L = \frac{N\Phi}{I} \text{ ,}$$

where in this case $N = 1$, and where $N >> r$, the inductance of the two-wire pair becomes

$$L = \frac{\mu_o}{\pi} \ln \frac{D}{r} \text{ Henrys per meter length.}$$

A round wire also has an internal inductance, which is separate from the external inductance shown above. The internal inductance is shown in electromagnetics texts to be

$$L_{int} = \frac{\mu_{int}}{8\pi} \text{ Henrys per meter length.}$$

For most current-carrying conductors, $\mu_{int} = \mu_o$ so that $L_{int} = 0.05\mu\text{H/m}$. Therefore, the total inductance of the two-wire circuit is the external inductance plus twice the internal inductance of each wire (i.e. current travels down and back), so that

$$L_{tot} = \frac{\mu_o}{\pi} \ln \frac{D}{r} + 2\frac{\mu_o}{8\pi} = \frac{\mu_o}{\pi}\left[\ln \frac{D}{r} + \frac{1}{4}\right] = \frac{\mu_o}{\pi}\left[\ln \frac{D}{r} + \ln\left(e^{\frac{1}{4}}\right)\right] = \frac{\mu_o}{\pi} \ln \frac{D}{re^{-\frac{1}{4}}} \ .$$

It is customary to define an effective radius

$$r_{eff} = re^{-\frac{1}{4}} = 0.7788r \ ,$$

and to write the total inductance in terms of it as

$$L_{tot} = \frac{\mu_o}{\pi} \ln \frac{D}{r_{eff}} \text{ Henrys per meter length.}$$

Wire Parallel to Earth's Surface

For a single wire of radius r, located at height h above the earth, the effect of the earth can be described by an image conductor, as it was for capacitance calculations. For a perfectly conducting earth, the image conductor is located h meters below the surface, as shown in Figure 3.17.

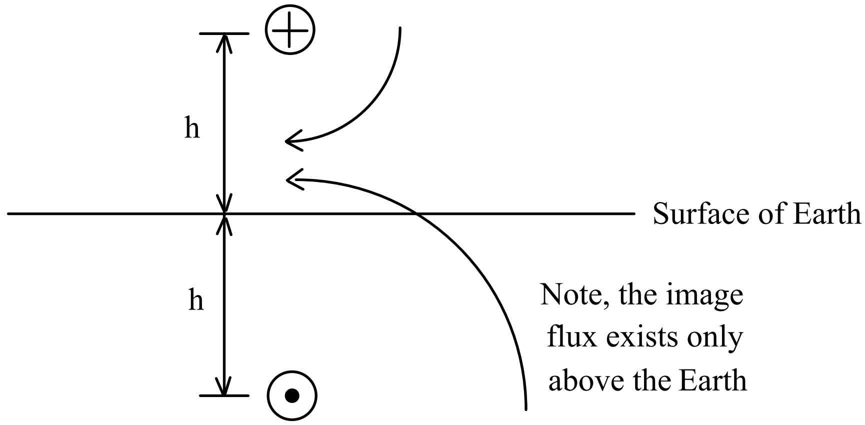


Figure 3.17: Current-Carrying Conductor Above Earth

The total flux linking the circuit is that which passes between the conductor and the surface of the earth. Summing the contribution of the conductor and its image yields

$$\Phi = \frac{\mu_o I}{2\pi}\left[\int_r^h \frac{dx}{x} + \int_h^{2h-r} \frac{dx}{x}\right] = \frac{\mu_o I}{2\pi}\ln\left[\frac{h(2h-r)}{rh}\right] = \frac{\mu_o I}{2\pi}\ln\left[\frac{(2h-r)}{r}\right] .$$

For $2h >> r$, a good approximation is

$$\Phi = \frac{\mu_o I}{2\pi}\ln\frac{2h}{r} \text{ Webers per meter length,}$$

so that the external inductance per meter length of the circuit becomes

$$L_{ext} = \frac{\mu_o}{2\pi}\ln\frac{2h}{r} \text{ Henrys per meter length.}$$

The total inductance is then the external inductance plus the internal inductance of one wire, or

$$L_{tot} = \frac{\mu_o}{2\pi}\ln\frac{2h}{r} + \frac{\mu_o}{8\pi} = \frac{\mu_o}{2\pi}\left[\ln\frac{2h}{r} + \frac{1}{4}\right] = \frac{\mu_o}{2\pi}\ln\frac{2h}{re^{-\frac{1}{4}}} ,$$

or, using the effective radius definition from before,

$$L_{tot} = \frac{\mu_o}{2\pi}\ln\frac{2h}{r_{eff}} \text{ Henrys per meter length.}$$

Bundled Conductors

The bundled conductor equivalent radii presented earlier apply for inductance as well as for capacitance. The question now is "what is the internal inductance of a bundle?" For *N* bundled conductors, the net internal inductance of a phase per meter must decrease as $\frac{1}{N}$ because the internal inductances are in parallel. Considering a bundle over the Earth, then

$$L_{tot} = \frac{\mu_o}{2\pi}\ln\frac{2h}{r_{eq}} + \frac{\mu_o}{8\pi N} = \frac{\mu_o}{2\pi}\left[\ln\frac{2h}{r_{eq}} + \frac{1}{4N}\right] = \frac{\mu_o}{2\pi}\left[\ln\frac{2h}{r_{eq}} + \frac{1}{N}\ln\left(e^{\frac{1}{4}}\right)\right] = \frac{\mu_o}{2\pi}\ln\left(\frac{2h}{r_{eq}e^{-\frac{1}{4N}}}\right).$$

Factoring in the expression for the equivalent bundle radius r_{eq} yields

$$r_{eq}e^{-\frac{1}{4N}} = \left[NrA^{N-1}\right]^{\frac{1}{N}} \bullet e^{-\frac{1}{4N}} = \left[Nre^{-\frac{1}{4}}A^{N-1}\right]^{\frac{1}{N}} = \left[Nr_{eff}A^{N-1}\right]^{\frac{1}{N}}$$

Thus, r_{eff} remains $re^{-\frac{1}{4}}$, no matter how many conductors are in the bundle.

The Three-Phase Case

For situations with multiples wires above the Earth, a matrix approach is needed. Consider the capacitance example given in Figure 3.11, except this time compute the external inductances, rather than capacitances. As far as the voltage (with respect to ground) of one of the a-b-c phases is concerned, the important flux is that which passes between the conductor and the Earth's surface. For example, the flux "linking" phase a will be produced by six currents: phase a current and its image, phase b current and its image, and phase c current and its image, and so on. Figure 3.18 is useful in visualizing the contribution of flux "linking" phase a that is caused by the current in phase b (and its image).

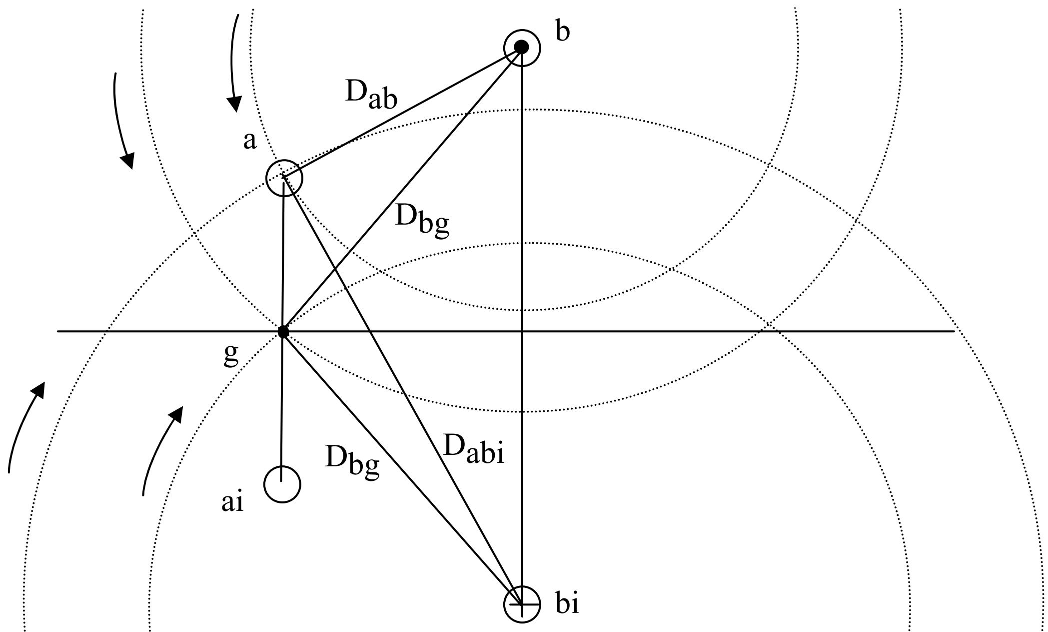


Figure 3.18. Flux Linking Phase a Due to Current in Phase b and Phase b Image

The linkage flux is

$$\Phi_a \text{ (due to } I_b \text{ and } I_b \text{ image)} = \frac{\mu_o I_b}{2\pi}\ln\frac{D_{bg}}{D_{ab}} + \frac{\mu_o I_b}{2\pi}\ln\frac{D_{abi}}{D_{bg}} = \frac{\mu_o I_b}{2\pi}\ln\frac{D_{abi}}{D_{ab}} .$$

Considering all phases, and applying superposition, yields the total flux

$$\Phi_a = \frac{\mu_o I_a}{2\pi}\ln\frac{D_{aai}}{r_a} + \frac{\mu_o I_b}{2\pi}\ln\frac{D_{abi}}{D_{ab}} + \frac{\mu_o I_c}{2\pi}\ln\frac{D_{aci}}{D_{ac}} .$$

Note that D_{aai} corresponds to 2h in Figure 3.15. Performing the same analysis for all three phases, and recognizing that $N\Phi = LI$, where N = 1 in this problem, then the inductance matrix is developed using

$$\begin{bmatrix} \Phi_a \\ \Phi_b \\ \Phi_c \end{bmatrix} = \frac{\mu_o}{2\pi}\begin{bmatrix} \ln\frac{D_{aai}}{r_a} & \ln\frac{D_{abi}}{D_{ab}} & \ln\frac{D_{aci}}{D_{ac}} \\ \ln\frac{D_{bai}}{D_{ba}} & \ln\frac{D_{bbi}}{r_b} & \ln\frac{D_{bci}}{D_{bc}} \\ \ln\frac{D_{cai}}{D_{ca}} & \ln\frac{D_{cbi}}{D_{cb}} & \ln\frac{D_{cci}}{r_c} \end{bmatrix}\begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix}, \text{ or } \Phi_{abc} = L_{abc} I_{abc} .$$

A comparison to the capacitance matrix derivation shows that the same matrix of natural logarithms is used in both cases, and that

$$L_{abc} = \frac{\mu_o}{2\pi} P_{abc} = \frac{\mu_o}{2\pi} \bullet 2\pi\varepsilon_o \bullet C_{abc}^{-1} = \mu_o \varepsilon C_{abc}^{-1} .$$

This implies that the product of the L and C matrices is a diagonal matrix with $\mu_o \varepsilon$ on the diagonal, providing that the earth is assumed to be a perfect conductor and that the internal inductances of the wires are ignored.

If the circuit has ground wires, then the dimension of L increases accordingly. Recognizing that the flux linking the ground wires is zero (because their voltages are zero), then L can be Kron reduced to yield an equivalent 3 x 3 matrix $L_{abc}^{'}$.

To include the internal inductance of the wires, replace actual conductor radius r with r_{eff}.

Computing 012 Inductances from Matrices

Once the 3 x 3 $L_{abc}^{'}$ matrix is found, 012 inductances can be determined by averaging the diagonal terms, and averaging the off-diagonal terms, of $L_{abc}^{'}$ to produce

$$L_{abc}^{avg} = \begin{bmatrix} L_S & L_M & L_M \\ L_M & L_S & L_S \\ L_M & L_M & L_S \end{bmatrix},$$

so that

$$L_{012}^{avg} = \begin{bmatrix} L_S + 2L_M & 0 & 0 \\ 0 & L_S - L_M & 0 \\ 0 & 0 & L_S - L_M \end{bmatrix}.$$

The Approximate Formulas for 012 Inductancess

Because of the similarity to the capacitance problem, the same rules for eliminating ground wires, for transposition, and for bundling conductors apply. Likewise, approximate formulas for the positive, negative, and zero sequence inductances can be developed, and these formulas are

$$L_1 = L_2 = \frac{\mu_o}{2\pi} \ln \frac{GMD_{1,2}}{GMR_{1,2}} ,$$

and

$$L_0 = 3\frac{\mu_o}{2\pi} \ln \frac{GMD_0}{GMR_0} .$$

It is important to note that the GMD and GMR terms for inductance differ from those of capacitance in two ways:

1. *GMR* calculations for inductance calculations should be made with $r_{eff} = re^{-\frac{1}{4}}$.

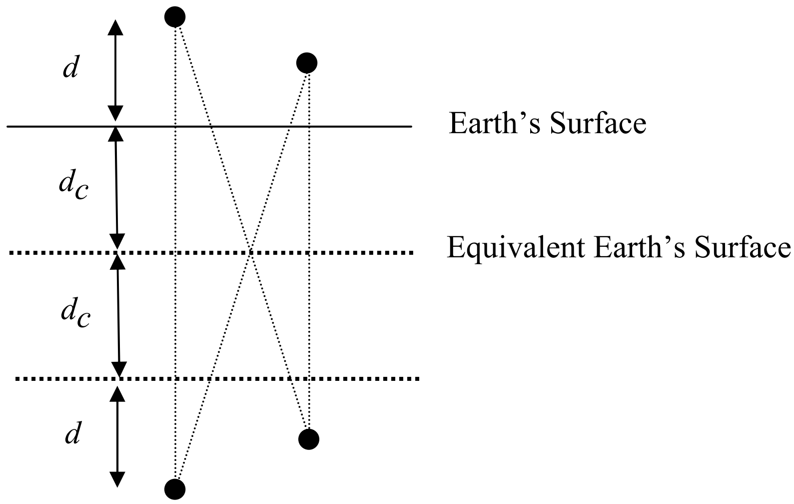
2. *GMD* distances for inductance calculations should include the equivalent complex depth for modeling finite conductivity earth (explained in the next section). This effect is ignored in capacitance calculations because the surface of the Earth is nominally at zero potential.

The Complex Depth Method for Modeling Imperfect Earth

The effect of the Earth's non-infinite conductivity should be included when computing inductances, especially zero sequence inductances. Because the Earth is not a perfect conductor, the image current does not actually flow on the surface of the Earth, but rather through a cross-section. The higher the conductivity, the narrower the cross-section.

The simplest method to account for finite conductivity is via a complex depth d_c , where the equivalent earth surface is assumed to be an additional d_c meters below the actual earth surface.

Using complex depth, if a conductor is d meters above earth, its image is $2(d + d_c)$ meters below the conductor.



The complex depth d_c is related to resistive skin depth d by

$$d_c = \frac{\delta}{1+j} = \frac{\delta}{\sqrt{2}}\angle -45^o = \frac{\delta}{2} - j\frac{\delta}{2} ,$$

where

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} .$$

At f = 60 Hz and $\sigma = 0.01\dfrac{1}{\Omega - m}$ (typical for limestone), the skin depth is 650 meters, so that

$$d_c = \frac{650}{2} - j\frac{650}{2} = 325 - j325 \text{ m.}$$

Skin depth in typical soils for 50 - 60 Hz varies from 500 to 2000 meters. The effect on positive/negative sequence inductances is not great, but the effect on zero sequence inductances is quite significant.

Since d_c is a frequency-dependent complex number, the self- and image-distance terms are complex, their natural logarithms are complex, and the L matrix contains complex numbers. The questions now are "how to take the natural log of a complex number and how to interpret the results?"

Begin with the natural log of a complex number $z = |z|e^{j\theta}$, which is

$$\ln(z) = \ln|z| + \ln(e^{j\vartheta}) = \ln|z| + j\theta \ , \ \theta \text{ in radians.}$$

Thus, for example, $\ln(325 - j325) = \ln(325) - j\dfrac{\pi}{4} = 5.78 - j0.785$.

To be consistent with the perfect Earth case (i.e., $\sigma \to \infty, \delta \to 0$), the real part of the natural logarithm must produce the inductive term. Then, the imaginary term yields the resistance of the Earth's current path, which must be added to the series resistances of the overhead conductors. In matrix form, the calculations are made accoding to

$$R_{abc}(\omega) + j\omega L_{abc}(\omega) = j\omega\frac{\mu_o}{2\pi}\begin{bmatrix} complex \ln\left(\dfrac{D_{aai}}{r_a}\right) & complex \ln\left(\dfrac{D_{abi}}{D_{ab}}\right) & complex \ln\left(\dfrac{D_{aci}}{D_{ac}}\right) \\ \bullet & complex \ln\left(\dfrac{D_{bbi}}{r_b}\right) & complex \ln\left(\dfrac{D_{bci}}{D_{bc}}\right) \\ \bullet & \bullet & complex \ln\left(\dfrac{D_{cci}}{r_c}\right) \end{bmatrix},$$

where complex depth d_c is included in the D terms. Because of the $j\omega$ multiplier, the real terms become positive, frequency-dependent resistances that account for the losses in the Earth. Off-diagonal resistances account for the fact that currents in neighboring phases contribute to the total voltage drop along the Earth, per meter, as seen by any given phase. The imaginary terms are inductive reactances.

3.2.3 Resistance and Conductance

The resistance of conductors is frequency dependent because of the resistive skin effect. Usually, however, this phenomenon is small for 50 - 60 Hz. Conductor resistances are readily obtained from tables, in the proper units of Ohms per meter length, and these values, added to the equivalent-earth resistances from the previous section, to yield the *R* used in the transmission line model.

Conductance *G* is very small for overhead transmission lines and can be ignored.

3.3 Underground Cables

Underground cables are transmission lines, and the model previously presented applies. Capacitance *C* tends to be much larger than for overhead lines, and conductance *G* should not be ignored.

For single-phase and three-phase cables, the capacitances and inductances per phase per meter length are

$$C = \frac{2\pi\varepsilon_o\varepsilon_r}{\ln\frac{b}{a}} \text{ Farads per meter length,}$$

and

$$L = \frac{\mu_o}{2\pi}\ln\frac{b}{a} \text{ Henrys per meter length,}$$

where b and a are the outer and inner radii of the coaxial cylinders. In power cables, $\frac{b}{a}$ is typically e (i.e., 2.7183) so that the voltage rating is maximized for a given diameter.

For most dielectrics, relative permittivity $\varepsilon_r = 2.0 - 2.5$. For three-phase situations, it is common to assume that the positive, negative, and zero sequence inductances and capacitances equal the above expressions. If the conductivity of the dielectric is known, conductance G can be calculated using

$$G = C\frac{\sigma}{\varepsilon} \text{ Mhos per meter length.}$$

3.2.5 Electric Field at Conductor Surface

Ignoring all other charges, the electric field at a conductor's surface can be approximated by

$$E_r = \frac{q}{2\pi\varepsilon_o r} ,$$

where r is the radius. For overhead conductors, this is a reasonable approximation because the neighboring line charges are relatively far away. It is always important to keep the peak electric field at a conductor's surface below 30kV/cm to avoid excessive corono losses.

Going beyond the above approximation, the Markt-Mengele method provides a detailed procedure for calculating the maximum peak subconductor surface electric field intensity for three-phase lines with identical phase bundles. Each bundle has N symmetric subconductors of radius r. The bundle radius is A. The procedure is

1. Treat each phase bundle as a single conductor with equivalent radius

$$r_{eq} = \left\lfloor NrA^{N-1} \right\rfloor^{1/N} .$$

2. Find the *C(N x N)* matrix, including ground wires, using average conductor heights above ground. Kron reduce *C(N x N)* to *C(3 x 3)*. Select the phase bundle that will have the greatest peak line charge value (q_{lpeak}) during a 60Hz cycle by successively placing maximum line-to-ground voltage V_{max} on one phase, and – V_{max}/2 on each of the other

two phases. Usually, the phase with the largest diagonal term in *C(3 by 3)* will have the greatest q_{lpeak}.

3. Assuming equal charge division on the phase bundle identified in Step 2, ignore equivalent line charge displacement, and calculate the average peak subconductor surface electric field intensity using

$$E_{avg,\,peak} = \frac{q_{lpeak}}{N} \bullet \frac{1}{2\pi\varepsilon_o r}$$

4. Take into account equivalent line charge displacement, and calculate the maximum peak subconductor surface electric field intensity using

$$E_{\max,\,peak} = E_{avg,\,peak}\left[1 + (N-1)\frac{r}{A}\right].$$